JOURNAL OF COMBINATORIAL THEORY 7, 342–352 (1969)

Solution of the Heawood Map-Coloring Problem—Case 2

GERHARD RINGEL AND J. W. T. YOUNGS*

Free University of Berlin, Berlin, Germany, and
University of California, Santa Cruz, California 95060

Received March 11, 1969

ABSTRACT

This paper gives a proof of the fact that the chromatic number of an orientable surface of genus p is equal to the integral part of $(7 + \sqrt{1 + 48p})/2$ whenever the latter is congruent to 2 modulo 12.

1. INTRODUCTION

In the interests of brevity the reader is referred to [3] for pertinent comments on the Heawood conjecture, and for the meanings of possibly unfamiliar terms and notations employed here.

Suffice it to say that the conjecture is that $\chi(S_p)$, the chromatic number of S_p, a surface of genus p, is given by the formula

$$\chi(S_p) = \left[\frac{7 + \sqrt{1 + 48p}}{2}\right] \quad \text{if} \quad p > 0. \tag{1}$$

The *Heawood theorem* is that $\chi(S_p)$ has the expression on the right of (1) as an upper bound.

An announcement of the complete solution was made in [1] and [2].

The usual method of attack is to consider $\gamma(K_n)$ the genus of the complete graph K_n.

The *complete graph theorem*,

$$\gamma(K_n) \geqslant \left\{\frac{(n-3)(n-4)}{12}\right\} \quad \text{for} \quad n \geqslant 3, \tag{2}$$

has been proved in many places (see for example [4]) and the *complete graph conjecture* is that equality holds in (2). For our purposes it is important to recall that, *if the complete graph conjecture is true, then the Heawood conjecture is confirmed* (see [1]).

* Both authors received partial support from the National Science Foundation.

The conjecture is proved in 12 cases depending upon the membership of n in the various residue classes modulo 12. If $n = 12s + k$ we say that we are dealing with Case k. In this paper we deal with Case 2.

Case 2 is not one of the regular cases (0, 3, 4, and 7) for which $(n-3)(n-4) \equiv 0 \pmod{12}$ and hence the Euler formula rules out the possibility of obtaining a triangular imbedding of K_{12s+2} (see [3]). We *modify* K_{12s+2} so that the Euler formula does not deny the possibility of obtaining a triangular imbedding of the modified graph.

If K is any graph without loops and multiple edges and is triangularly imbedded in a surface S then we know

$$6\gamma(K) = 6\gamma(S) = 6 - 3\alpha_0 + \alpha_1 , \tag{3}$$

where α_0 is the number of vertices and α_1 the number of arcs in K (see [3]).

In Case 2, the natural approach is to consider the graph $K_n - K_2$, which is obtained from K_n by removing one arc. Then (3) becomes

$$\gamma(K_n - K_2) = \gamma(S) = \frac{(n-2)(n-5)}{12}$$

and $(n-2)(n-5) \equiv 0 \pmod{12}$ if $n \equiv 2 \pmod{12}$. Hence, an apparently appropriate modification of K_{12s+2} is to remove one arc from it. This approach is successful if s is odd (see [2]) and a triangular imbedding of $K_{24t+14} - K_2$ has been obtained for $t \geqslant 0$. After much labor we abandoned the above modification of K_{12s+2} if s is even and cast about for something new.

Consider a graph $L_{12\sigma+2}$ defined as follows. There are $12\sigma + 3$ vertices denoted by 0, 1, 2,..., $12\sigma - 7$, a, b, c, u, v, w, x, y_0, y_1. Each pair of vertices identified by numbers is joined by an arc; the vertices a, b, c, u, v, w, and x are joined to all the numbered vertices; the vertex y_0 is joined to all the *even* numbered vertices, y_1 to all the *odd*. Note that there are no arcs connecting vertices which are identified by letters.

The graph of $L_{12\sigma+2}$ appears to be very dissimilar to $K_{12\sigma+2}$ but a comparison can be made as follows. If y_0 and y_1 are identified so as to obtain a single vertex y, then we obtain the graph $K_{12\sigma+2} - K_8$, that is, $K_{12\sigma+2}$ with the 28 arcs joining the vertices a, b, c, u, v, w, x, and y of $K_{12\sigma+2}$ removed.

In using (3) on $L_{12\sigma+2}$ note that $\alpha_0 = 12\sigma + 3$ and $\alpha_1 = 72\sigma^2 + 18\sigma - 27$, hence

$$\gamma(L_{12\sigma+2}) = 12\sigma^2 - 3\sigma - 5.$$

Consequently the Euler formula does not rule out the possibility of a triangular imbedding of $L_{12\sigma+2}$ in a surface S of genus $12\sigma^2 - 3\sigma - 5$.

We shall obtain such an imbedding as a solution of the *regular* part of the problem.

In reference to the comparison between $K_{12\sigma+2}$ and $L_{12\sigma+2}$ note that, by (2),

$$\gamma(K_{12\sigma+2}) \geqslant 12\sigma^2 - 3\sigma + 1. \tag{4}$$

Hence, if we can accommodate the identification of y_0 and y_1 together with the missing 28 arcs of K_8 without adding more than 6 handles to S, we shall have

$$\gamma(K_{12\sigma+2}) \leqslant 12\sigma^2 - 3\sigma + 1. \tag{5}$$

Hence (4) and (5) will prove the complete graph conjecture in Case 2.

Modification of S so as to obtain an imbedding of $K_{12\sigma+2}$ in a new surface of genus $\gamma(S) + 6$ is the *irregular* or *additional adjacency* part of the problem.

The interrelation between the regular and irregular parts of such problems has been emphasized in [3]. In this case the combinatorial requirements of the irregular part impose conditions on the regular part found in the displayed portions of Figures 2 and 3.

The combinatorial requirements of the irregular part were different from those here presented when Case 2 was initially solved by these methods in early October 1967. The initial combinatorial requirements led to a solution of the regular problem in 3 subcases. The requirements presented here were developed toward the end of April 1968 and were a happy discovery, since they enable us to use the solution of the corresponding regular problem in Case 11 (see [3]).

2. Illustration of the Method

As an example consider the diagram in Figure 1, where the currents are from Z_{30}.

It is helpful to compare Figure 1 with Figure 2 of [3]. The central parts of the two diagrams are *identical*, the only changes are in the first three

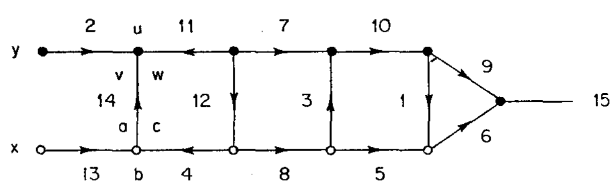

FIGURE 1

arcs at the left and the arcs in the triangle on the right. In this case there are four vortices: x, y, abc, and uvw. In common with Figure 2 of [3]:

I. Each element $1,\ldots,15$ of Z_{30} *appears exactly once and* 15, *the element of order* 2 *appears on the singular arc.*

The rotations are different in this case; nevertheless

II. *The rotations induce a single circuit.*

Row 0 of the scheme given by reading the successive currents on the circuit is

$$\begin{array}{l} 15\ 24\ 29\ 20\ 27\ 22\ 18\ 11\ w\ 16\ a\ 17\ x\ \ 13\ b\ 26\ 8\ \ 5 \\ \ 6\ 21\ \ 1\ 25\ \ 3\ 23\ 12\ \ 4\ c\ \ 14\ v\ 28\ y_0\ \ 2\ u\ 19\ 7\ 10\ 9. \end{array}$$

This is entirely analogous to row 0 obtained from Figure 2 of [3] except that, where one would have recorded y above, we here record y_0. The reason for this will become clear in a few lines.

The numerical part of row i is obtained by adding i modulo 30 to the elements of Z_{30} in row 0. In regard to the placement of the letters in the rows identified by elements of Z_{30}, the positions occupied by x and y are unchanged; however, y is replaced by y_0 in the even rows, and by y_1 in the odd rows. The letters a, b, c and u, v, w are permuted as below

$$\begin{array}{lllllllllllllllllll}
0. & \cdots 11 & w & 16 & a & 17 & x & 13 & b & 26 \cdots 4 & c & 14 & v & 28 & y_0 & 2 & u & 19 \cdots \\
1. & \cdots 12 & u & 17 & b & 18 & x & 14 & c & 27 \cdots 5 & a & 15 & w & 29 & y_1 & 3 & v & 20 \cdots \\
2. & \cdots 13 & v & 18 & c & 19 & x & 15 & a & 28 \cdots 6 & b & 16 & u & 0 & y_0 & 4 & w & 21 \cdots \\
3. & \cdots 14 & w & 19 & a & 20 & x & 16 & b & 29 \cdots 7 & c & 17 & v & 1 & y_1 & 5 & u & 22 \cdots
\end{array}$$

etc.

The rows identified by letters are determined by the requirement that R^* be fulfilled (see [3]) and this implies that

$$\begin{array}{lrrrrrrcrrr}
x. & 17 & 4 & 21 & . & . & . & \ldots & 26 & 13 & 0 \\
y_0. & 28 & 26 & 24 & . & . & . & \ldots & 4 & 2 & 0 \\
y_1. & 29 & 27 & 25 & . & . & . & \ldots & 5 & 3 & 1 \\
a. & 17 & 0 & 16 & 20 & 3 & 19 & \ldots & 14 & 27 & 13 \\
b. & 26 & 0 & 13 & 29 & 3 & 16 & \ldots & 23 & 27 & 10 \\
c. & 14 & 0 & 4 & 17 & 3 & 7 & \ldots & 11 & 27 & 1 \\
u. & 19 & 0 & 2 & 16 & 27 & 29 & \ldots & 22 & 3 & 5 \\
v. & 28 & 0 & 14 & 26 & 27 & 11 & \ldots & 1 & 3 & 17 \\
w. & 16 & 0 & 11 & 13 & 27 & 8 & \ldots & 19 & 3 & 14
\end{array}$$

It is important to observe that *every* number but *no* letter appears in all the rows identified by letters except the two rows y_0 and y_1. Row y_0 is a permutation of the even numbers in Z_{30}, row y_1 the odd. This is so because

III. *The current flowing out of* x, in this case 13, *generates* Z_{30}, *and the current flowing out of* y, in this case 2, *generates the subgroup of* Z_{30} *consisting of even elements.*

IV. *The currents flowing into* $abc(uvw)$, in this case 4, 13, and -14 (2, 11, and 14) *are congruent to each other but not congruent to* 0 *modulo* 3, *and their sum,* 3 ($27 = -3$), *generates the subgroup of* Z_{30} *consisting of all elements divisible by* 3.

On the other hand, because of I and II, in row i every number and all the letters a, b, c, u, v, w, x, and y_j with $j \equiv i \pmod 2$ are found.

Hence the complete scheme is for the graph L_{38}. The proof that the *scheme is triangular* proceeds exactly as the illustration in [3] using the basic facts that:

V. *Each vertex in Figure* 1 *other than* x *and* y *has valence* 3.

VI. *Kirchhoff's Current Law is satisfied at each vertex of valence* 3 *except the two vortices* abc *and* uvw.

Hence we have obtained a triangular scheme for L_{38} and thus a triangular imbedding of L_{38} in an orientable surface S.

3. The General Solution of the Regular Problem

We wish to find a triangular imbedding of $L_{12\sigma+2}$ with $\sigma = s + 1$ in an orientable surface, that is, find a triangular scheme for $L_{12\sigma+2}$.

Generalizing on the example for $s = 2$ we use one of the ladder-like diagrams (Figures 2 and 3) in which the currents are from Z_{12s+6}. In each case there are $2s$ rungs.

Note that V of the last section is satisfied. Where rotations are not specified place • on the upper horizontal and ○ on the lower. It is easy to see that II is fulfilled with a very regular circuit.

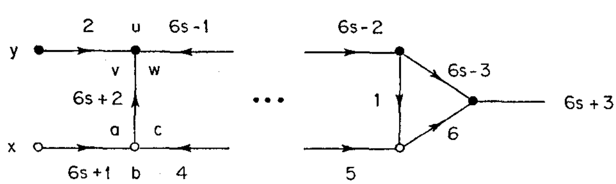

Figure 2

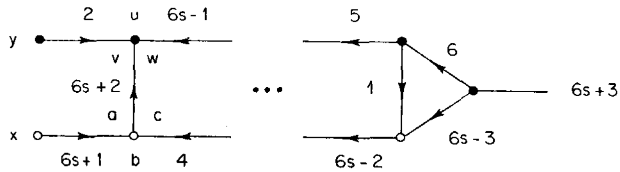

FIGURE 3

The reader should compare these figures with Figures 4 and 5 of [3]. The number of rungs in all figures is $2s$. In [3] the current pairs P_0 and P_{2s} are on the displayed parts of the diagram and the same is the case here. The current 1 of P_0 is found on the last rung, and $6s + 2$ on the first; the currents 2 and $6s + 1$ of P_{2s} are found on the first paired arcs at the extreme left.

Most important of all, the left-most pair in the central part of *all* diagrams is P_1 while the right-most is P_{2s-1}. In [3] we required that the alternating sequences end with the triple $P_2\ 2s - 3\ P_{2s-1}$; here no such requirement is made.

Suppose s is even. Take the *central part* of the solution to the distribution problem in [3], a solution which will use Figure 4 of [3], and transfer it to Figure 2. The analogous transfer is made from Figure 5 to Figure 3 if s is odd. The completed diagrams satisfy I and VI:

Each element $1,..., 6s + 3$ of Z_{12s+6} appears exactly once and $6s + 3$, the element of order 2, is on the singular arc.

Kirchhoff's Current Law holds at each vertex of valence 3 except the two vortices *abc* and *uvw*.

Let us now examine the vortices. The current $6s + 1$ which flows out of x generates Z_{12s+6}, while the current 2 flowing out of y generates the subgroup of even elements Z_{12s+6}.

The currents leading into *abc* are 4, $6s + 1$, $-(6s + 2)$, all congruent to 1 modulo 3 and their sum is 3 which generates the subgroup of elements of Z_{12s+6} divisible by 3. The currents 2, $6s - 1$, and $6s + 2$ leading into *uvw* are all congruent to 2 modulo 3 and their sum -3 generates the same subgroup.

Hence all the properties I to VI of the special example L_{38} are satisfied and exactly as in that case give a triangular scheme for $L_{12(s+1)+2}$ and thus a triangular imbedding of $L_{12(s+1)+2}$ in a surface S.

In fact we obtain a solution of the regular part also for $s = 1$. Figure 3 is compressed and there are no undisplayed rungs. Our general solution obviously does not apply to the case $s = 0$. Here we rely on the imbedding of K_{14} obtained in [2].

4. The Additional Adjacency Problem

Consider the dual decomposition of the triangular imbedding of $L_{12(s+1)+2}$ obtained in the preceding section. Regard this as a map on S in which each 2-cell is a country identified by the vertex of $L_{12(s+1)+2}$ in its interior.

All the pertinent information for the additional adjacency problem is contained on the left side of Figure 2 (which is identical with the left side of Figure 3).

In the complete scheme, that portion of row 0 given by the left side is

$$\begin{array}{l} 0. \quad \cdots\ 6s-1\ \ w\ \ 6s+4\ \ a\ \ 6s+5\ \ x\ \ 6s+1\ \ b\ \ 12s+2\ \cdots \\ \qquad\quad \cdots 4\ \ c\ \ 6s+2\ \ v\ \ 12s+4\ \ y_0\ \ \ 2\ \ \ u\ \ 6s+7\ \cdots \end{array}$$

Consider also the following part of the scheme:

$$\begin{array}{llcll} 0. & \cdots\ 6s+2 & v & 12s+4 & \cdots \\ 1. & \cdots\ 6s+3 & w & 12s+5 & \cdots \\ 2. & \cdots\ 6s+4 & u & 0 & \cdots \end{array}$$

The above information gives a partial picture of the map on S around countries 0 and 2 found in Figure 4.

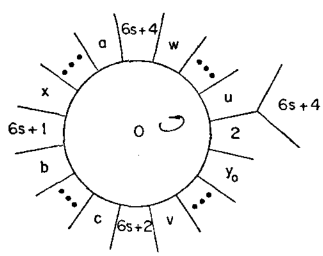

FIGURE 4

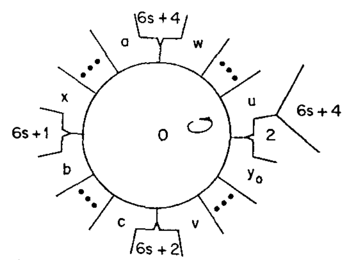

FIGURE 5

First modification. We modify the map on S as illustrated in Figure 5. We *gain* the adjacencies

$$(a, w), (b, x), (c, v), (u, y_0) \tag{6}$$

but we lose the adjacencies

$$(0, 2), (0, 6s+1), (0, 6s+2), (0, 6s+4). \tag{7}$$

First handle. Consider the map on a torus shown in Figure 6. Go to Figure 5 and excise the country 0. Do the same with the country z from Figure 6. Identify the boundaries of the two resulting surfaces in the

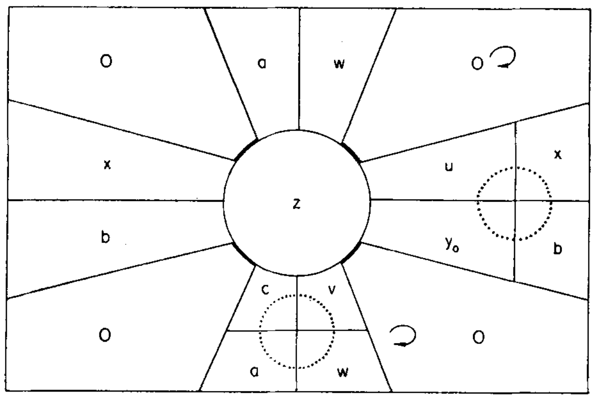

FIGURE 6

obvious way. After this there is a new country, named 0, that is adjacent to the same countries as the old country 0 in Figure 5. Notice that we have gained the adjacencies

$$(a, c), (b, y_0), (x, u), (v, w). \tag{8}$$

Second handle. From Figure 6 excise the interiors of the two dotted circles and identify the boundaries as in Figure 7. Note that the identification is made so that the new surface is orientable—the same will be true in identifications to follow.

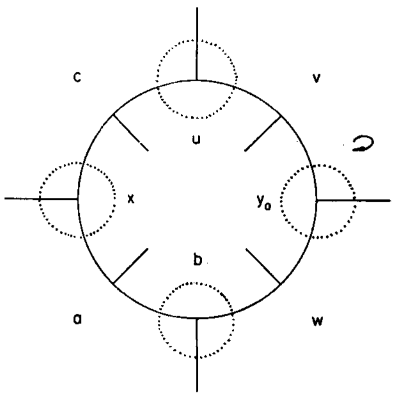

FIGURE 7

We gain the adjacencies

$$(a, b), (b, w), (w, y_0), (y_0, v), (v, u), (u, c), (c, x), (x, a). \tag{9}$$

Third handle. Excise the interiors of the upper and lower dotted circles in Figure 7. Identify the boundaries of these circles as in Figure 8.

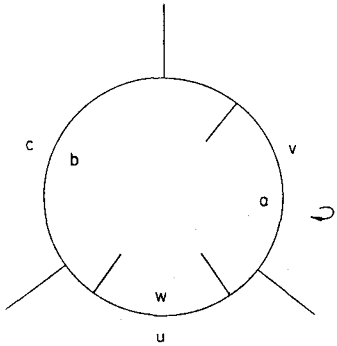

FIGURE 8

We gain the adjacencies

$$(a, u), (u, b), (b, v), (v, a), (b, c), (u, w). \tag{10}$$

Fourth handle. Excise the interiors of the dotted circles to the left and right of Figure 7. Identify the boundaries of these circles as in Figure 9.

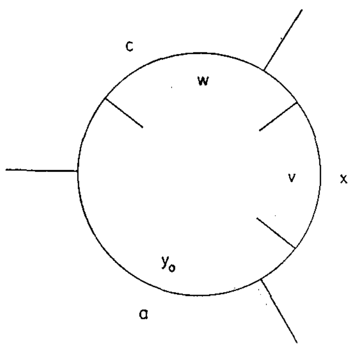

FIGURE 9

We gain the adjacencies

$$(c, y_0), (y_0, x), (x, w), (w, c), (a, y_0), (v, x). \tag{11}$$

It is interesting to stop and survey the present state of affairs. We have, with the addition of four handles, obtained all 28 adjacencies between the countries a, b, c, u, v, w, x, y_0. On the other hand, we have lost the four adjacencies of (7).

Fifth handle. From Figure 5 and the fact that u must have a contact with $6s + 2$ we obtain the information about u shown in Figure 10. Consider the map on the torus shown in Figure 11. Excise country u from Figure 10 and z from Figure 11 and identify the boundaries on the obvious way.

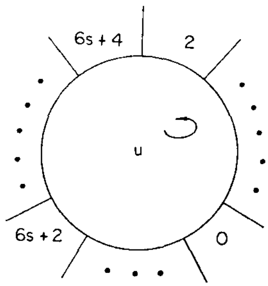

FIGURE 10

There is a new country u which has all the adjacencies of the old country u of Figure 10. This handle gains us the adjacencies

$$(0, 2), (0, 6s + 2), (0, 6s + 4). \tag{12}$$

Hence we have recovered three of adjacencies (7) lost in the first modification.

Sixth handle. To complete the additional adjacency problem we need to recover the remaining lost adjacency $(0, 6s + 1)$ and identify y_0 and y_1. Of crucial importance is the fact that $6s + 1$ is odd, therefore adjacent

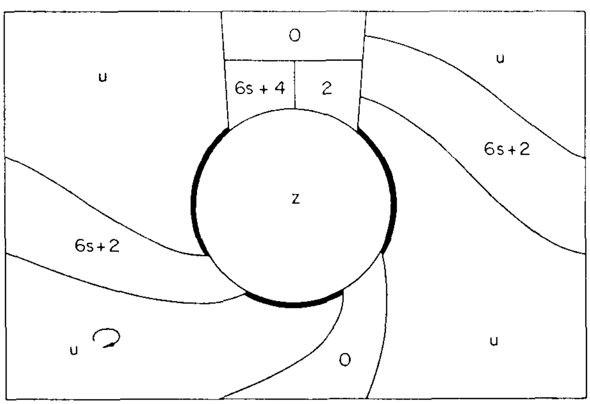

FIGURE 11

to y_1, while 0 is adjacent to y_0. It is obvious that, by the addition of a handle, we can simultaneously obtain an adjacency $(0, 6s + 1)$ and an adjacency (y_0, y_1). Erase the common frontier between y_0 and y_1, and call the new country y.

Our final map consists of $12(s + 1) + 2 = 12\sigma + 2$ countries each adjacent to all the others. The dual therefore contains $K_{12\sigma+2}$ as a subgraph. This shows the complete graph conjecture to be true in Case 2 for $\sigma \geqslant 2$.

References

1. G. Ringel and J. W. T. Youngs, Solution of the Heawood Map-Coloring Problem, *Proc. Nat. Acad. Sci. U.S.A.* **60** (1968), 438–445.
2. G. Ringel and J. W. T. Youngs, Lösung des Problems der Nachbargebiete auf orientierbaren Flächen, *Archiv der Mathematik* **20** (1969), 190–201.
3. G. Ringel and J. W. T. Youngs, Solution of the Heawood Map-Coloring Problem—Case 11, *J. Combinatorial Theory* **7** (1969), 71–93.
4. J. W. T. Youngs, The Heawood Map-Coloring Conjecture, Chapter 12 in *Graph Theory and Theoretical Physics* (F. Harary, ed.), Academic Press, New York-London, 1967, pp. 313–354.